Mathematical Physics

Individual Contest

To complete this assessment, answer at least 3 of the following 4 problems.

Problem 1

A non-relativistic, spinless, quantum particle of mass m moves in the Euclidean plane $\mathbb{R}^2$. The particle is connected by *ideal* springs of elastic constant k to three *fixed* points in the plane $\{p_1, p_2, p_3\} \subset \mathbb{R}^2$ which lay at the vertices of an equilateral triangle whose sides have length L.

(a) determine the Lie group of continuous symmetries of the system (the generators of its Lie algebra should contain a maximal set of functional independent conserved quantities);

(b) compute the energy levels of the quantum system.

Problem 2

Let Γ_v be the Caley tree of valency $v \geq 3$, i.e. the connected graph over infinitely-many nodes without non-trivial closed paths ($\equiv$ a tree) where each node is connected to other v nodes by a single link. On each node $i \in \Gamma_v$ there is a Ising spin σ_i which takes the two values ± 1 (spin *up* and spin *down*) and the system has Hamiltonian ($\equiv$ energy)

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \quad J > 0$$

where the notation $\sum_{\langle i,j \rangle}$ means that the sum is over pairs of sites $i, j \in \Gamma_v$ connected by a link ($\equiv$ nearest neighbors sites).

Question. Use the free boundary condition. Is there a *positive* temperature T_c at which we have a phase transition and yet the free energy $F(T)$ is real analytic for $0 < T < \infty$?

Problem 3

Orders-of-magnitude black hole physics.

Assuming black holes indeed evaporate via quantum effect as predicted by Hawking. Answer the following questions to an order of magnitude. (Hint: for this problem, it is the most convenient to work in the Planck units with $c = G = \hbar = k_B = 1$ and convert the answer in the end using the values of Planck length, mass, time, and temperature: $l_P \sim 10^{-35}$ m, $m_P \sim 10^{-8}$ kg, $t_P \sim 10^{-43}$ s, and $T_P \sim 10^{32}$ K. Also, please feel free to ignore 2's and π's.)

1. If a black hole has mass M and its Hawking radiation has a black-body spectrum with temperature proportional to its "surface gravity" (at, say, twice the radius of the horizon), what is the temperature and total power of the Hawking radiation in Planck units? And for a solar-mass black hole ($\sim 10^{38}\, m_P$), what is the temperature and power in Kelvins and Watts?

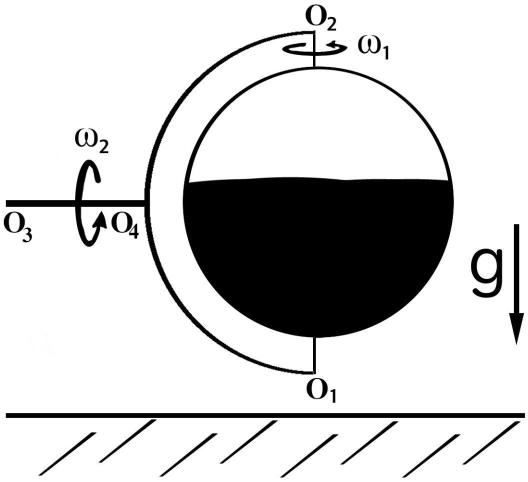

Figure 1: Problem 4

2. What is the maximum mass of a black hole that can evaporate via Hawking radiation in the lifetime of the Universe (10 billion years $\sim 10^{60} t_P$)? Compare its temperature with that of the cosmic microwave background ($\sim 2.7K$).

3. How much mass (in kilograms) does a black hole radiate away in its final second? Would this be large or small in astronomical standard?

4. For a solar-mass black hole, what is the charge in conventional units (Coulombs) that would have to be present to make $Q = M$ in relativistic units? How does this compare to the total charge of a solar mass of protons? (Hint: the charge of a proton is 10^{-19} Coulomb while its mass is $10^{-19} m_P$.)

Problem 4

A spherical container is partially filled with a slowly hardening epoxy resin (Figure 1). The container is spinning with angular velocity ω_1 about the $O_1 - O_2$ axis, which in turn is spinning with angular velocity ω_2 about the fixed $O_3 - O_4$ axis. Assuming $\omega_1 \neq \omega_2$, find the final shape of the hardened resin (use g for gravitational acceleration at Earth's surface).